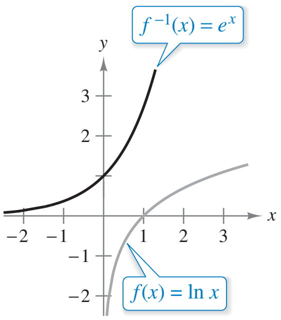
5.4 Exponential Functions: Differentiation and Integration

- **Develop properties of the natural exponential function.**
- **Differentiate natural exponential functions.**
- **Integrate natural exponential functions.**

The Natural Exponential Function

The function $f(x) = \ln x$ is increasing on its entire domain, and therefore it has an inverse function f^{-1}. The domain of f^{-1} is the set of all real numbers, and the range is the set of positive real numbers, as shown in Figure 5.19. So, for any real number x,

$$f(f^{-1}(x)) = \ln[f^{-1}(x)] = x. \qquad x \text{ is any real number.}$$



The inverse function of the natural logarithmic function is the natural exponential function.
Figure 5.19

If x is rational, then

$$\ln(e^x) = x \ln e = x(1) = x. \qquad x \text{ is a rational number.}$$

Because the natural logarithmic function is one-to-one, you can conclude that $f^{-1}(x)$ and e^x agree for *rational* values of x. The next definition extends the meaning of e^x to include *all* real values of x.

Definition of the Natural Exponential Function

The inverse function of the natural logarithmic function $f(x) = \ln x$ is called the **natural exponential function** and is denoted by

$$f^{-1}(x) = e^x.$$

That is,

$$y = e^x \quad \text{if and only if} \quad x = \ln y.$$

THE NUMBER e

The symbol e was first used by mathematician Leonhard Euler to represent the base of natural logarithms in a letter to another mathematician, Christian Goldbach, in 1731.

The inverse relationship between the natural logarithmic function and the natural exponential function can be summarized as shown.

$$\ln(e^x) = x \quad \text{and} \quad e^{\ln x} = x \qquad \text{Inverse relationship}$$

EXAMPLE 1 Solving an Exponential Equation

Solve $7 = e^{x+1}$.

Solution You can convert from exponential form to logarithmic form by *taking the natural logarithm of each side* of the equation.

$$7 = e^{x+1} \qquad \text{Write original equation.}$$
$$\ln 7 = \ln(e^{x+1}) \qquad \text{Take natural logarithm of each side.}$$
$$\ln 7 = x + 1 \qquad \text{Apply inverse property.}$$
$$-1 + \ln 7 = x \qquad \text{Solve for } x.$$

So, the solution is $-1 + \ln 7 \approx -0.946$. You can check this solution as shown.

$$7 = e^{x+1} \qquad \text{Write original equation.}$$
$$7 \stackrel{?}{=} e^{(-1+\ln 7)+1} \qquad \text{Substitute } -1 + \ln 7 \text{ for } x \text{ in original equation.}$$
$$7 \stackrel{?}{=} e^{\ln 7} \qquad \text{Simplify.}$$
$$7 = 7 \checkmark \qquad \text{Solution checks.}$$

EXAMPLE 2 **Solving a Logarithmic Equation**

Solve $\ln(2x - 3) = 5$.

Solution To convert from logarithmic form to exponential form, you can *exponentiate each side* of the logarithmic equation.

$$\ln(2x - 3) = 5 \quad \text{Write original equation.}$$
$$e^{\ln(2x-3)} = e^5 \quad \text{Exponentiate each side.}$$
$$2x - 3 = e^5 \quad \text{Apply inverse property.}$$
$$x = \tfrac{1}{2}(e^5 + 3) \quad \text{Solve for } x.$$
$$x \approx 75.707 \quad \text{Use a calculator.}$$

The familiar rules for operating with rational exponents can be extended to the natural exponential function, as shown in the next theorem.

THEOREM 5.10 Operations with Exponential Functions

Let a and b be any real numbers.

1. $e^a e^b = e^{a+b}$ **2.** $\dfrac{e^a}{e^b} = e^{a-b}$

Proof To prove Property 1, you can write

$$\ln(e^a e^b) = \ln(e^a) + \ln(e^b) = a + b = \ln(e^{a+b}).$$

Because the natural logarithmic function is one-to-one, you can conclude that

$$e^a e^b = e^{a+b}.$$

The proof of the other property is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

In Section 5.3, you learned that an inverse function f^{-1} shares many properties with f. So, the natural exponential function inherits the properties listed below from the natural logarithmic function.

Properties of the Natural Exponential Function

1. The domain of $f(x) = e^x$ is

$$(-\infty, \infty)$$

and the range is

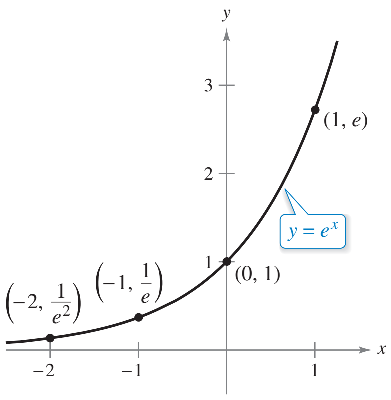
$$(0, \infty).$$

2. The function $f(x) = e^x$ is continuous, increasing, and one-to-one on its entire domain.

3. The graph of $f(x) = e^x$ is concave upward on its entire domain.

4. $\lim_{x \to -\infty} e^x = 0$

5. $\lim_{x \to \infty} e^x = \infty$



The natural exponential function is increasing, and its graph is concave upward.

Derivatives of Exponential Functions

One of the most intriguing (and useful) characteristics of the natural exponential function is that *it is its own derivative*. In other words, it is a solution of the differential equation $y' = y$. This result is stated in the next theorem.

REMARK You can interpret this theorem geometrically by saying that the slope of the graph of $f(x) = e^x$ at any point (x, e^x) is equal to the y-coordinate of the point.

THEOREM 5.11 Derivatives of the Natural Exponential Function

Let u be a differentiable function of x.

1. $\dfrac{d}{dx}[e^x] = e^x$
2. $\dfrac{d}{dx}[e^u] = e^u \dfrac{du}{dx}$

Proof To prove Property 1, use the fact that $\ln e^x = x$, and differentiate each side of the equation.

$$\ln e^x = x \qquad \text{Definition of exponential function}$$

$$\frac{d}{dx}[\ln e^x] = \frac{d}{dx}[x] \qquad \text{Differentiate each side with respect to } x.$$

$$\frac{1}{e^x}\frac{d}{dx}[e^x] = 1$$

$$\frac{d}{dx}[e^x] = e^x$$

The derivative of e^u follows from the Chain Rule.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

FOR FURTHER INFORMATION To find out about derivatives of exponential functions of order 1/2, see the article "A Child's Garden of Fractional Derivatives" by Marcia Kleinz and Thomas J. Osler in *The College Mathematics Journal*. To view this article, go to *MathArticles.com.*

EXAMPLE 3 Differentiating Exponential Functions

Find the derivative of each function.

a. $y = e^{2x-1}$ **b.** $y = e^{-3/x}$

Solution

a. $\dfrac{d}{dx}[e^{2x-1}] = e^u\dfrac{du}{dx} = 2e^{2x-1}$ $\qquad u = 2x - 1$

b. $\dfrac{d}{dx}[e^{-3/x}] = e^u\dfrac{du}{dx} = \left(\dfrac{3}{x^2}\right)e^{-3/x} = \dfrac{3e^{-3/x}}{x^2}$ $\qquad u = -\dfrac{3}{x}$

EXAMPLE 4 Locating Relative Extrema

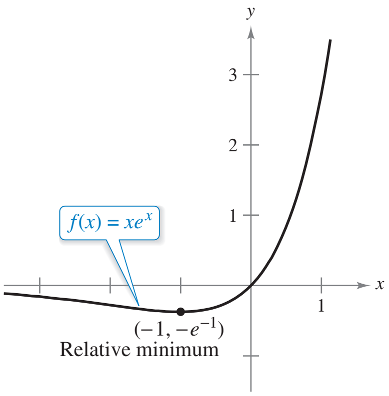
Find the relative extrema of

$$f(x) = xe^x.$$

Solution The derivative of f is

$$f'(x) = x(e^x) + e^x(1) \qquad \text{Product Rule}$$
$$= e^x(x + 1).$$

Because e^x is never 0, the derivative is 0 only when $x = -1$. Moreover, by the First Derivative Test, you can determine that this corresponds to a relative minimum, as shown in Figure 5.20. Because the derivative $f'(x) = e^x(x + 1)$ is defined for all x, there are no other critical points.



The derivative of f changes from negative to positive at $x = -1$.
Figure 5.20

EXAMPLE 5 The Standard Normal Probability Density Function

See LarsonCalculus.com for an interactive version of this type of example.

Show that the *standard normal probability density function*

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

has points of inflection when $x = \pm 1$.

Solution To locate possible points of inflection, find the x-values for which the second derivative is 0.

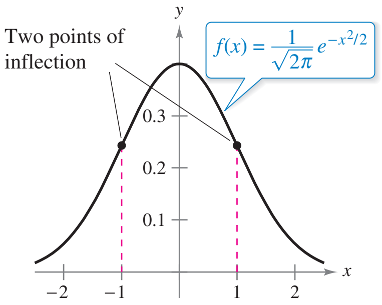
$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad \text{Write original function.}$$

$$f'(x) = \frac{1}{\sqrt{2\pi}}(-x)e^{-x^2/2} \quad \text{First derivative}$$

$$f''(x) = \frac{1}{\sqrt{2\pi}}\left[(-x)(-x)e^{-x^2/2} + (-1)e^{-x^2/2}\right] \quad \text{Product Rule}$$

$$= \frac{1}{\sqrt{2\pi}}(e^{-x^2/2})(x^2 - 1) \quad \text{Second derivative}$$

So, $f''(x) = 0$ when $x = \pm 1$, and you can apply the techniques of Chapter 3 to conclude that these values yield the two points of inflection shown in Figure 5.21.



The bell-shaped curve given by a standard normal probability density function
Figure 5.21

REMARK The general form of a normal probability density function (whose mean is 0) is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/(2\sigma^2)}$$

where σ is the standard deviation (σ is the lowercase Greek letter sigma). This "bell-shaped curve" has points of inflection when $x = \pm\sigma$.

EXAMPLE 6 Population of California

The projected populations y (in thousands) of California from 2015 through 2030 can be modeled by

$$y = 34{,}696e^{0.0097t}$$

where t represents the year, with $t = 15$ corresponding to 2015. At what rate will the population be changing in 2020? (*Source: U.S. Census Bureau*)

Solution The derivative of the model is

$$y' = (0.0097)(34{,}696)e^{0.0097t}$$

$$\approx 336.55e^{0.0097t}.$$

By evaluating the derivative when $t = 20$, you can estimate that the rate of change in 2020 will be about

408.6 thousand people per year.

Integrals of Exponential Functions

Each differentiation formula in Theorem 5.11 has a corresponding integration formula.

> **THEOREM 5.12 Integration Rules for Exponential Functions**
>
> Let u be a differentiable function of x.
>
> **1.** $\int e^x\,dx = e^x + C$ **2.** $\int e^u\,du = e^u + C$

EXAMPLE 7 Integrating Exponential Functions

Find the indefinite integral.

$$\int e^{3x+1}\,dx$$

Solution If you let $u = 3x + 1$, then $du = 3\,dx$.

$$\begin{aligned}
\int e^{3x+1}\,dx &= \frac{1}{3}\int e^{3x+1}(3)\,dx && \text{Multiply and divide by 3.}\\
&= \frac{1}{3}\int e^u\,du && \text{Substitute: } u = 3x + 1.\\
&= \frac{1}{3}e^u + C && \text{Apply Exponential Rule.}\\
&= \frac{e^{3x+1}}{3} + C && \text{Back-substitute.}
\end{aligned}$$

REMARK In Example 7, the missing *constant* factor 3 was introduced to create $du = 3\,dx$. However, remember that you cannot introduce a missing *variable* factor in the integrand. For instance,

$$\int e^{-x^2}\,dx \neq \frac{1}{x}\int e^{-x^2}(x\,dx).$$

EXAMPLE 8 Integrating Exponential Functions

Find the indefinite integral.

$$\int 5xe^{-x^2}\,dx$$

Solution If you let $u = -x^2$, then $du = -2x\,dx$ or $x\,dx = -du/2$.

$$\begin{aligned}
\int 5xe^{-x^2}\,dx &= \int 5e^{-x^2}(x\,dx) && \text{Regroup integrand.}\\
&= \int 5e^u\left(-\frac{du}{2}\right) && \text{Substitute: } u = -x^2.\\
&= -\frac{5}{2}\int e^u\,du && \text{Constant Multiple Rule}\\
&= -\frac{5}{2}e^u + C && \text{Apply Exponential Rule.}\\
&= -\frac{5}{2}e^{-x^2} + C && \text{Back-substitute.}
\end{aligned}$$

EXAMPLE 9 Integrating Exponential Functions

Find each indefinite integral.

a. $\displaystyle\int \frac{e^{1/x}}{x^2}\,dx$ **b.** $\displaystyle\int \sin x\, e^{\cos x}\,dx$

Solution

a. $$\int \frac{e^{1/x}}{x^2}\,dx = -\int \overbrace{e^{1/x}}^{e^u}\overbrace{\left(-\frac{1}{x^2}\right)dx}^{du} \qquad u = \frac{1}{x}$$
$$= -e^{1/x} + C$$

b. $$\int \sin x\, e^{\cos x}\,dx = -\int \overbrace{e^{\cos x}}^{e^u}\overbrace{(-\sin x\,dx)}^{du} \qquad u = \cos x$$
$$= -e^{\cos x} + C$$

EXAMPLE 10 Finding Areas Bounded by Exponential Functions

Evaluate each definite integral.

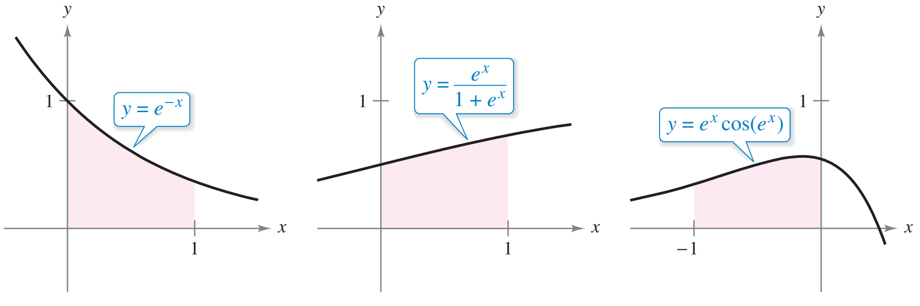
a. $\displaystyle\int_0^1 e^{-x}\,dx$ **b.** $\displaystyle\int_0^1 \frac{e^x}{1+e^x}\,dx$ **c.** $\displaystyle\int_{-1}^0 [e^x\cos(e^x)]\,dx$

Solution

a. $$\int_0^1 e^{-x}\,dx = -e^{-x}\Big]_0^1 \qquad \text{See Figure 5.22(a).}$$
$$= -e^{-1} - (-1)$$
$$= 1 - \frac{1}{e}$$
$$\approx 0.632$$

b. $$\int_0^1 \frac{e^x}{1+e^x}\,dx = \ln(1+e^x)\Big]_0^1 \qquad \text{See Figure 5.22(b).}$$
$$= \ln(1+e) - \ln 2$$
$$\approx 0.620$$

c. $$\int_{-1}^0 [e^x\cos(e^x)]\,dx = \sin(e^x)\Big]_{-1}^0 \qquad \text{See Figure 5.22(c).}$$
$$= \sin 1 - \sin(e^{-1})$$
$$\approx 0.482$$



(a) (b) (c)

Figure 5.22

5.4 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Solving an Exponential or Logarithmic Equation **In Exercises 1–16, solve for x accurate to three decimal places.**

1. $e^{\ln x} = 4$
2. $e^{\ln 3x} = 24$
3. $e^x = 12$
4. $5e^x = 36$
5. $9 - 2e^x = 7$
6. $8e^x - 12 = 7$
7. $50e^{-x} = 30$
8. $100e^{-2x} = 35$
9. $\dfrac{800}{100 - e^{x/2}} = 50$
10. $\dfrac{5000}{1 + e^{2x}} = 2$
11. $\ln x = 2$
12. $\ln x^2 = 10$
13. $\ln(x - 3) = 2$
14. $\ln 4x = 1$
15. $\ln\sqrt{x + 2} = 1$
16. $\ln(x - 2)^2 = 12$

Sketching a Graph **In Exercises 17–22, sketch the graph of the function.**

17. $y = e^{-x}$
18. $y = \frac{1}{2}e^x$
19. $y = e^x + 2$
20. $y = e^{x-1}$
21. $y = e^{-x^2}$
22. $y = e^{-x/2}$

23. **Comparing Graphs** Use a graphing utility to graph $f(x) = e^x$ and the given function in the same viewing window. How are the two graphs related?

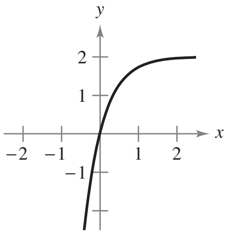
(a) $g(x) = e^{x-2}$ (b) $h(x) = -\frac{1}{2}e^x$ (c) $q(x) = e^{-x} + 3$

24. **Asymptotes** Use a graphing utility to graph the function. Use the graph to determine any asymptotes of the function.

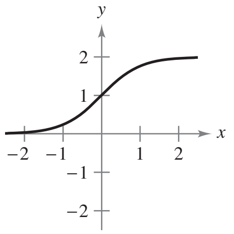
(a) $f(x) = \dfrac{8}{1 + e^{-0.5x}}$ (b) $g(x) = \dfrac{8}{1 + e^{-0.5/x}}$

Matching **In Exercises 25–28, match the equation with the correct graph. Assume that a and C are positive real numbers. [The graphs are labeled (a), (b), (c), and (d).]**

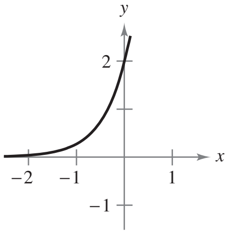
(a)



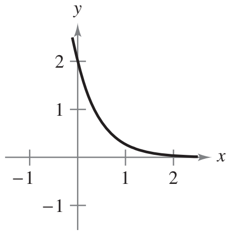
(b)



(c)



(d)



25. $y = Ce^{ax}$
26. $y = Ce^{-ax}$
27. $y = C(1 - e^{-ax})$
28. $y = \dfrac{C}{1 + e^{-ax}}$

Inverse Functions **In Exercises 29–32, illustrate that the functions are inverses of each other by graphing both functions on the same set of coordinate axes.**

29. $f(x) = e^{2x}$, $g(x) = \ln\sqrt{x}$
30. $f(x) = e^{x/3}$, $g(x) = \ln x^3$
31. $f(x) = e^x - 1$, $g(x) = \ln(x + 1)$
32. $f(x) = e^{x-1}$, $g(x) = 1 + \ln x$

Finding a Derivative **In Exercises 33–54, find the derivative.**

33. $f(x) = e^{2x}$
34. $y = e^{-8x}$
35. $y = e^{\sqrt{x}}$
36. $y = e^{-2x^3}$
37. $y = e^{x-4}$
38. $y = 5e^{x^2+5}$
39. $y = e^x \ln x$
40. $y = xe^{4x}$
41. $y = x^3e^x$
42. $y = x^2e^{-x}$
43. $g(t) = (e^{-t} + e^t)^3$
44. $g(t) = e^{-3/t^2}$
45. $y = \ln(1 + e^{2x})$
46. $y = \ln\left(\dfrac{1 + e^x}{1 - e^x}\right)$
47. $y = \dfrac{2}{e^x + e^{-x}}$
48. $y = \dfrac{e^x - e^{-x}}{2}$
49. $y = \dfrac{e^x + 1}{e^x - 1}$
50. $y = \dfrac{e^{2x}}{e^{2x} + 1}$
51. $y = e^x(\sin x + \cos x)$
52. $y = e^{2x}\tan 2x$
53. $F(x) = \displaystyle\int_{\pi}^{\ln x} \cos e^t\, dt$
54. $F(x) = \displaystyle\int_0^{e^{2x}} \ln(t + 1)\, dt$

Finding an Equation of a Tangent Line **In Exercises 55–62, find an equation of the tangent line to the graph of the function at the given point.**

55. $f(x) = e^{3x}$, $(0, 1)$
56. $f(x) = e^{-2x}$, $(0, 1)$
57. $f(x) = e^{1-x}$, $(1, 1)$
58. $y = e^{-2x+x^2}$, $(2, 1)$
59. $f(x) = e^{-x}\ln x$, $(1, 0)$
60. $y = \ln\dfrac{e^x + e^{-x}}{2}$, $(0, 0)$
61. $y = x^2e^x - 2xe^x + 2e^x$, $(1, e)$
62. $y = xe^x - e^x$, $(1, 0)$

Implicit Differentiation **In Exercises 63 and 64, use implicit differentiation to find dy/dx.**

63. $xe^y - 10x + 3y = 0$
64. $e^{xy} + x^2 - y^2 = 10$

Finding the Equation of a Tangent Line **In Exercises 65 and 66, find an equation of the tangent line to the graph of the function at the given point.**

65. $xe^y + ye^x = 1$, $(0, 1)$
66. $1 + \ln xy = e^{x-y}$, $(1, 1)$

Finding a Second Derivative In Exercises 67 and 68, find the second derivative of the function.

67. $f(x) = (3 + 2x)e^{-3x}$

68. $g(x) = \sqrt{x} + e^x \ln x$

Differential Equation In Exercises 69 and 70, show that the function $y = f(x)$ is a solution of the differential equation.

69. $y = 4e^{-x}$
$y'' - y = 0$

70. $y = e^{3x} + e^{-3x}$
$y'' - 9y = 0$

Finding Extrema and Points of Inflection In Exercises 71–78, find the extrema and the points of inflection (if any exist) of the function. Use a graphing utility to graph the function and confirm your results.

71. $f(x) = \dfrac{e^x + e^{-x}}{2}$

72. $f(x) = \dfrac{e^x - e^{-x}}{2}$

73. $g(x) = \dfrac{1}{\sqrt{2\pi}}e^{-(x-2)^2/2}$

74. $g(x) = \dfrac{1}{\sqrt{2\pi}}e^{-(x-3)^2/2}$

75. $f(x) = x^2e^{-x}$

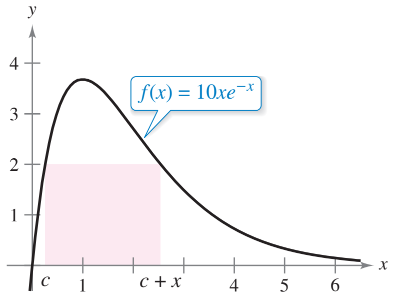
76. $f(x) = xe^{-x}$

77. $g(t) = 1 + (2 + t)e^{-t}$

78. $f(x) = -2 + e^{3x}(4 - 2x)$

79. Area Find the area of the largest rectangle that can be inscribed under the curve $y = e^{-x^2}$ in the first and second quadrants.

80. Area Perform the following steps to find the maximum area of the rectangle shown in the figure.



(a) Solve for c in the equation $f(c) = f(c + x)$.

(b) Use the result in part (a) to write the area A as a function of x. [*Hint:* $A = xf(c)$]

(c) Use a graphing utility to graph the area function. Use the graph to approximate the dimensions of the rectangle of maximum area. Determine the maximum area.

(d) Use a graphing utility to graph the expression for c found in part (a). Use the graph to approximate

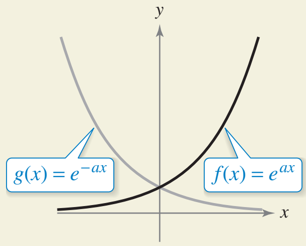
$$\lim_{x\to 0^+} c \quad \text{and} \quad \lim_{x\to\infty} c.$$

Use this result to describe the changes in dimensions and position of the rectangle for $0 < x < \infty$.

81. Finding an Equation of a Tangent Line Find a point on the graph of the function $f(x) = e^{2x}$ such that the tangent line to the graph at that point passes through the origin. Use a graphing utility to graph f and the tangent line in the same viewing window.

Robert Adrian Hillman/Shutterstock.com

82. HOW DO YOU SEE IT? The figure shows the graphs of f and g, where a is a positive real number. Identify the open interval(s) on which the graphs of f and g are (a) increasing or decreasing, and (b) concave upward or concave downward.



83. Depreciation The value V of an item t years after it is purchased is $V = 15{,}000e^{-0.6286t}$, $0 \le t \le 10$.

(a) Use a graphing utility to graph the function.

(b) Find the rates of change of V with respect to t when $t = 1$ and $t = 5$.

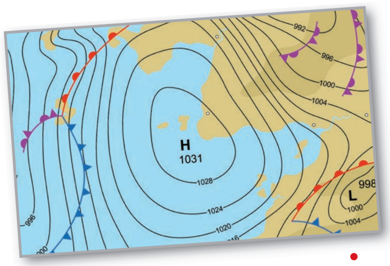
(c) Use a graphing utility to graph the tangent lines to the function when $t = 1$ and $t = 5$.

84. Harmonic Motion The displacement from equilibrium of a mass oscillating on the end of a spring suspended from a ceiling is $y = 1.56e^{-0.22t}\cos 4.9t$, where y is the displacement (in feet) and t is the time (in seconds). Use a graphing utility to graph the displacement function on the interval $[0, 10]$. Find a value of t past which the displacement is less than 3 inches from equilibrium.

85. Atmospheric Pressure

A meteorologist measures the atmospheric pressure P (in kilograms per square meter) at altitude h (in kilometers). The data are shown below.

h	0	5	10	15	20
P	10,332	5583	2376	1240	517



(a) Use a graphing utility to plot the points $(h, \ln P)$. Use the regression capabilities of the graphing utility to find a linear model for the revised data points.

(b) The line in part (a) has the form $\ln P = ah + b$. Write the equation in exponential form.

(c) Use a graphing utility to plot the original data and graph the exponential model in part (b).

(d) Find the rate of change of the pressure when $h = 5$ and $h = 18$.

86. Modeling Data The table lists the approximate values V of a mid-sized sedan for the years 2006 through 2012. The variable t represents the time (in years), with $t = 6$ corresponding to 2006.

t	6	7	8	9
V	\$23,046	\$20,596	\$18,851	\$17,001

t	10	11	12
V	\$15,226	\$14,101	\$12,841

(a) Use the regression capabilities of a graphing utility to fit linear and quadratic models to the data. Plot the data and graph the models.

(b) What does the slope represent in the linear model in part (a)?

(c) Use the regression capabilities of a graphing utility to fit an exponential model to the data.

(d) Determine the horizontal asymptote of the exponential model found in part (c). Interpret its meaning in the context of the problem.

(e) Use the exponential model to find the rate of decrease in the value of the sedan when $t = 7$ and $t = 11$.

Linear and Quadratic Approximation **In Exercises 87 and 88, use a graphing utility to graph the function. Then graph**

$$P_1(x) = f(0) + f'(0)(x - 0) \quad \text{and}$$

$$P_2(x) = f(0) + f'(0)(x - 0) + \tfrac{1}{2}f''(0)(x - 0)^2$$

in the same viewing window. Compare the values of f, P_1, P_2, and their first derivatives at $x = 0$.

87. $f(x) = e^x$ **88.** $f(x) = e^{x/2}$

Stirling's Formula **For large values of n,**

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdot \cdot (n - 1) \cdot n$$

can be approximated by Stirling's Formula,

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$$

In Exercises 89 and 90, find the exact value of $n!$, and then approximate $n!$ using Stirling's Formula.

89. $n = 12$ **90.** $n = 15$

Finding an Indefinite Integral **In Exercises 91–108, find the indefinite integral.**

91. $\int e^{5x}(5)\,dx$ **92.** $\int e^{-x^4}(-4x^3)\,dx$

93. $\int e^{2x-1}\,dx$ **94.** $\int e^{1-3x}\,dx$

95. $\int x^2 e^{x^3}\,dx$ **96.** $\int e^x(e^x + 1)^2\,dx$

97. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$ **98.** $\int \frac{e^{1/x^2}}{x^3}\,dx$

99. $\int \frac{e^{-x}}{1 + e^{-x}}\,dx$ **100.** $\int \frac{e^{2x}}{1 + e^{2x}}\,dx$

101. $\int e^x\sqrt{1 - e^x}\,dx$ **102.** $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$

103. $\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx$ **104.** $\int \frac{2e^x - 2e^{-x}}{(e^x + e^{-x})^2}\,dx$

105. $\int \frac{5 - e^x}{e^{2x}}\,dx$ **106.** $\int \frac{e^{2x} + 2e^x + 1}{e^x}\,dx$

107. $\int e^{-x}\tan(e^{-x})\,dx$

108. $\int e^{2x}\csc(e^{2x})\,dx$

Evaluating a Definite Integral **In Exercises 109–118, evaluate the definite integral. Use a graphing utility to verify your result.**

109. $\int_0^1 e^{-2x}\,dx$ **110.** $\int_1^2 e^{5x-3}\,dx$

111. $\int_0^1 xe^{-x^2}\,dx$ **112.** $\int_{-2}^0 x^2 e^{x^3/2}\,dx$

113. $\int_1^3 \frac{e^{3/x}}{x^2}\,dx$ **114.** $\int_0^{\sqrt{2}} xe^{-(x^2/2)}\,dx$

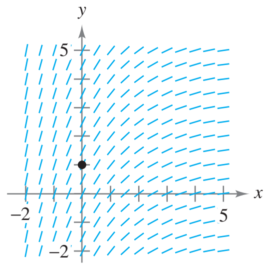
115. $\int_0^3 \frac{2e^{2x}}{1 + e^{2x}}\,dx$ **116.** $\int_0^1 \frac{e^x}{5 - e^x}\,dx$

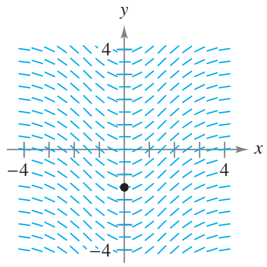
117. $\int_0^{\pi/2} e^{\sin \pi x}\cos \pi x\,dx$

118. $\int_{\pi/3}^{\pi/2} e^{\sec 2x}\sec 2x \tan 2x\,dx$

Slope Field **In Exercises 119 and 120, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a). To print an enlarged copy of the graph, go to *MathGraphs.com*.**

119. $\frac{dy}{dx} = 2e^{-x/2}$, $(0, 1)$ **120.** $\frac{dy}{dx} = xe^{-0.2x^2}$, $\left(0, -\frac{3}{2}\right)$





Differential Equation **In Exercises 121 and 122, solve the differential equation.**

121. $\frac{dy}{dx} = xe^{ax^2}$ **122.** $\frac{dy}{dx} = (e^x - e^{-x})^2$

Differential Equation In Exercises 123 and 124, find the particular solution that satisfies the initial conditions.

123. $f''(x) = \frac{1}{2}(e^x + e^{-x})$, $f(0) = 1, f'(0) = 0$

124. $f''(x) = \sin x + e^{2x}$, $f(0) = \frac{1}{4}, f'(0) = \frac{1}{2}$

Area In Exercises 125–128, find the area of the region bounded by the graphs of the equations. Use a graphing utility to graph the region and verify your result.

125. $y = e^x, y = 0, x = 0, x = 5$

126. $y = e^{-2x}, y = 0, x = -1, x = 3$

127. $y = xe^{-x^2/4}, y = 0, x = 0, x = \sqrt{6}$

128. $y = e^{-2x} + 2, y = 0, x = 0, x = 2$

Numerical Integration In Exercises 129 and 130, approximate the integral using the Midpoint Rule, the Trapezoidal Rule, and Simpson's Rule with $n = 12$. Use a graphing utility to verify your results.

129. $\displaystyle\int_0^4 \sqrt{x}\, e^x\, dx$

130. $\displaystyle\int_0^2 2xe^{-x}\, dx$

131. Probability A car battery has an average lifetime of 48 months with a standard deviation of 6 months. The battery lives are normally distributed. The probability that a given battery will last between 48 months and 60 months is

$$0.0065 \int_{48}^{60} e^{-0.0139(t-48)^2}\, dt.$$

Use the integration capabilities of a graphing utility to approximate the integral. Interpret the resulting probability.

132. Probability The median waiting time (in minutes) for people waiting for service in a convenience store is given by the solution of the equation

$$\int_0^x 0.3e^{-0.3t}\, dt = \frac{1}{2}.$$

What is the median waiting time?

133. Using the Area of a Region Find the value of a such that the area bounded by $y = e^{-x}$, the x-axis, $x = -a$, and $x = a$ is $\frac{8}{3}$.

134. Modeling Data A valve on a storage tank is opened for 4 hours to release a chemical in a manufacturing process. The flow rate R (in liters per hour) at time t (in hours) is given in the table.

t	0	1	2	3	4
R	425	240	118	71	36

(a) Use the regression capabilities of a graphing utility to find a linear model for the points $(t, \ln R)$. Write the resulting equation of the form $\ln R = at + b$ in exponential form.

(b) Use a graphing utility to plot the data and graph the exponential model.

(c) Use the definite integral to approximate the number of liters of chemical released during the 4 hours.

WRITING ABOUT CONCEPTS

135. Properties of the Natural Exponential Function In your own words, state the properties of the natural exponential function.

136. A Function and Its Derivative Is there a function f such that $f(x) = f'(x)$? If so, identify it.

137. Choosing a Function Without integrating, state the integration formula you can use to integrate each of the following.

(a) $\displaystyle\int \frac{e^x}{e^x + 1}\, dx$

(b) $\displaystyle\int xe^{x^2}\, dx$

138. Analyzing a Graph Consider the function

$$f(x) = \frac{2}{1 + e^{1/x}}.$$

(a) Use a graphing utility to graph f.

(b) Write a short paragraph explaining why the graph has a horizontal asymptote at $y = 1$ and why the function has a nonremovable discontinuity at $x = 0$.

139. Deriving an Inequality Given $e^x \geq 1$ for $x \geq 0$, it follows that

$$\int_0^x e^t\, dt \geq \int_0^x 1\, dt.$$

Perform this integration to derive the inequality

$e^x \geq 1 + x$

for $x \geq 0$.

140. Solving an Equation Find, to three decimal places, the value of x such that $e^{-x} = x$. (Use Newton's Method or the *zero* or *root* feature of a graphing utility.)

141. Horizontal Motion The position function of a particle moving along the x-axis is $x(t) = Ae^{kt} + Be^{-kt}$, where A, B, and k are positive constants.

(a) During what times t is the particle closest to the origin?

(b) Show that the acceleration of the particle is proportional to the position of the particle. What is the constant of proportionality?

142. Analyzing a Function Let $f(x) = \dfrac{\ln x}{x}$.

(a) Graph f on $(0, \infty)$ and show that f is strictly decreasing on (e, ∞).

(b) Show that if $e \leq A < B$, then $A^B > B^A$.

(c) Use part (b) to show that $e^\pi > \pi^e$.

143. Finding the Maximum Rate of Change Verify that the function

$$y = \frac{L}{1 + ae^{-x/b}}, \quad a > 0, \quad b > 0, \quad L > 0$$

increases at a maximum rate when $y = L/2$.